

THE \mathcal{A} -DECOMPOSABILITY OF THE SINGER CONSTRUCTION

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ABSTRACT. Let $R_s M$ denote the Singer construction on an unstable module M over the Steenrod algebra \mathcal{A} at the prime two; $R_s M$ is canonically a subobject of $P_s \otimes M$, where $P_s = \mathbb{F}_2[x_1, \dots, x_s]$ with generators of degree one. Passage to \mathcal{A} -indecomposables gives the natural transformation $R_s M \rightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes M)$, which identifies with the dual of the composition of the Singer transfer and the Lannes-Zarati homomorphism.

The main result of the paper proves the weak generalized algebraic spherical class conjecture, which was proposed by the first named author. Namely, this morphism is trivial on elements of positive degree when $s > 2$. The condition $s > 2$ is necessary, as exhibited by the spherical classes of Hopf invariant one and those of Kervaire invariant one.

1. INTRODUCTION

The Hurewicz map

$$(1) \quad \pi_*(\Omega^\infty \Sigma^\infty X) \rightarrow H_*(\Omega^\infty \Sigma^\infty X; \mathbb{F}_2)$$

from the stable homotopy groups of a pointed space X to the mod 2 homology of the infinite loop space $QX := \Omega^\infty \Sigma^\infty X$ is of significant interest; for example, for $X = S^0$, so that $\pi_*(QS^0)$ gives the stable homotopy of the sphere spectrum, the spherical class conjecture predicts that the image in positive degree consists only of the images of elements of Hopf invariant one and those of Kervaire invariant one (see [Hun97], [Cur75] and [Wel82], for example). This conjecture is usually attributed either to Madsen or to Curtis.

The first named author (cf. [HT15, Conjecture 1.1]) proposed the following generalization (which is related to a conjecture due to Peter Eccles cited in [Zar09]). Write $Q_0 X$ for the basepoint component of QX and henceforth always take homology (and cohomology) with \mathbb{F}_2 coefficients.

Conjecture 1 (The generalized spherical class conjecture). Let X be a pointed CW-complex. Then the Hurewicz homomorphism $H : \pi_*(Q_0 X) \rightarrow H_*(Q_0 X)$ vanishes on classes of $\pi_*(Q_0 X)$ of Adams filtration greater than 2.

This paper is motivated by an algebraic version of the generalized spherical class conjecture, stated below as Conjecture 2; the main result, Theorem 1, proves a weak algebraic version, Conjecture 3. These conjectures are due to the first named author; to state them, we recall the Singer transfer and the Lannes-Zarati morphism.

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An algebraic model for the $\mathbb{Z}/2$ -transfer $\Sigma^\infty \mathbb{R}P_+^\infty \rightarrow \Sigma^\infty S^0$ (see [MMM86], for example) is given by the non-trivial extension class $e \in \text{Ext}_{\mathcal{A}}^1(\Sigma^{-1}\mathbb{F}_2, P_1)$, where $P_1 = H^*(\mathbb{R}P^\infty) \cong \mathbb{F}_2[x]$. This class is represented by the short exact sequence of \mathcal{A} -modules

$$0 \longrightarrow P_1 \longrightarrow \widehat{P}_1 \longrightarrow \Sigma^{-1}\mathbb{F}_2 \longrightarrow 0,$$

where \widehat{P}_1 denotes the submodule of elements of degree ≥ -1 in the algebra $\mathbb{F}_2[x^{\pm 1}]$ equipped with the structure of \mathcal{A} -algebra extending that on P_1 .

For a positive integer s , the s -fold iterated smash product of the transfer is represented by the class $e^s \in \text{Ext}_{\mathcal{A}}^s(\Sigma^{-s}\mathbb{F}_2, P_s)$, given by the tensor product of s copies of e , where $P_s \cong H^*(BV_s) \cong P_1^{\otimes s}$.

Recall that the destabilization functor \mathcal{D} from \mathcal{A} -modules to unstable modules gives $\mathcal{D}N$, the largest unstable quotient of the \mathcal{A} -module N . There is a natural transformation $\mathcal{D}N \rightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} N$ for functors from \mathcal{A} -modules to \mathcal{A} -modules, where $\mathbb{F}_2 \otimes_{\mathcal{A}} N$ has trivial \mathcal{A} -module structure. This is obtained by applying \mathcal{D} to the quotient $N \rightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} N$ and then composing with the canonical inclusion:

$$\mathcal{D}N \rightarrow \mathcal{D}(\mathbb{F}_2 \otimes_{\mathcal{A}} N) \equiv (\mathbb{F}_2 \otimes_{\mathcal{A}} N)^{\geq 0} \xrightarrow{\subseteq} \mathbb{F}_2 \otimes_{\mathcal{A}} N.$$

This passes to derived functors to give

$$\mathcal{D}_s N \rightarrow \text{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, N).$$

The cap product with the algebraic transfer class e^s induces a natural transformation $\cap e^s : \mathcal{D}_s(\Sigma^{-s}N) \rightarrow \mathcal{D}(P_s \otimes N)$ for any \mathcal{A} -module N and hence, for M an unstable module,

$$\alpha_s^M : \mathcal{D}_s(\Sigma^{-s}M) \rightarrow P_s \otimes M.$$

Lannes and Zarati [LZ87] used this to relate the functors \mathcal{D}_s to the Singer functors R_s . (As recalled in Section 2, the Singer functor R_s is an exact functor defined on the category of unstable modules and $R_s M$ is canonically a submodule of $P_s \otimes M$, for M an unstable module.) Namely, by [LZ87, Théorème 2.5], for M an unstable module, $\alpha_s^{\Sigma M}$ induces an isomorphism

$$\mathcal{D}_s(\Sigma^{1-s}M) \xrightarrow{\cong} \Sigma R_s M.$$

Hence, as in [LZ87], there is a natural homomorphism of \mathcal{A} -modules $\Sigma R_s M \rightarrow \text{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{1-s}M) \cong \Sigma \text{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-s}M)$ and thus

$$\mathbb{F}_2 \otimes_{\mathcal{A}} R_s M \rightarrow \text{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-s}M).$$

The linear dual of this map:

$$(2) \quad \text{Ext}_{\mathcal{A}}^s(\Sigma^{-s}M, \mathbb{F}_2) \rightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} R_s M)^*$$

is called the Lannes-Zarati homomorphism. It corresponds to an associated graded of the Hurewicz map (1) when M is the reduced cohomology of a pointed space X . (The proof of this assertion is sketched in [Lan88] and [Goe86]; the Singer functors arise in identifying $H^*(QX)$ in terms of $H^*(X)$.)

The first named author proposed the following algebraic version of the generalized spherical class conjecture for $M = \mathbb{F}_2$ in [Hun97] and, for any unstable module M , in his seminar at the Vietnam National University (cf. [HT15, Conjecture 1.2]).

Conjecture 2 (The generalized algebraic spherical class conjecture). The Lannes-Zarati homomorphism (2) vanishes in positive stem, for $s > 2$ and any unstable \mathcal{A} -module M .

The conjecture was established by the first named author and his co-workers for the case $M = \mathbb{F}_2$ for $s = 3, 4$, and 5 respectively in [Hun97] and [Hun99], [Hun03], and [HQT14]. That the Lannes-Zarati morphism for $M = \mathbb{F}_2$ vanishes for $s > 2$ on the algebra decomposables of $\text{Ext}_{\mathcal{A}}^s(\mathbb{F}_2, \mathbb{F}_2)$ was proved in [HP98].

The definition of Singer's algebraic transfer [Sin89] is similar to that of the Lannes-Zarati morphism, replacing the derived functors of destabilization by $\text{Tor}_s^{\mathcal{A}}$. Namely, for an \mathcal{A} -module N , the cap product with e^s induces a natural transformation:

$$\psi_s^N : \text{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-s}N) \rightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes N).$$

The dual of this map is called the Singer transfer. By naturality of the cap product, this fits into the commutative diagram:

$$\begin{array}{ccc} \mathcal{D}_s(\Sigma^{-s}N) & \xrightarrow{\cap e^s} & \mathcal{D}(P_s \otimes N) \\ \downarrow & & \downarrow \\ \text{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-s}N) & \xrightarrow{\psi_s^N} & \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes N) \end{array}$$

in which the vertical morphisms are the natural transformations.

In particular, for $N = \Sigma M$, where M is an unstable module, the dual of the composition of the Singer transfer with the Lannes-Zarati morphism identifies (up to suspension) with the composite:

$$R_s M \hookrightarrow P_s \otimes M \twoheadrightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes M)$$

of the canonical inclusion with the projection to \mathcal{A} -indecomposables.

A weak form of Conjecture 2 asserts that the s -th Lannes-Zarati homomorphism vanishes for $s > 2$ in any positive stem on the image of the Singer transfer. This is equivalent to the following (see also [HT15, Conjecture 1.6]):

Conjecture 3 (The weak generalized algebraic spherical class conjecture). Let M be an unstable \mathcal{A} -module and $s > 2$ be an integer. Then every positive degree element of the Singer construction $R_s M$ is \mathcal{A} -decomposable in $P_s \otimes M$.

The conjecture was proved for $M = \tilde{H}^*(S^0)$ and $M = \tilde{H}^*(BV_k)$ (for k a non-negative integer) by Trần N. Nam and the first named author in [HN01a] and [HN01b] respectively.

Conjecture 3 is proved by the main result of the paper:

Theorem 1. *If $s > 2$, the morphism*

$$R_s M \rightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes M)$$

is trivial on elements of positive degree, for any unstable \mathcal{A} -module M .

The condition $s > 2$ is necessary: in the case $M = \mathbb{F}_2$, the map is non-zero for $s \in \{1, 2\}$ (see [Hun97]). This phenomenon arises from the existence of the spherical classes of Hopf invariant one and those of Kervaire invariant one.

Example 1.1.

- (i) Taking $M = \mathbb{F}_2$ recovers the main result of [HN01a];
- (ii) taking $M = P_k$ for k a non-negative integer recovers the main result of [HN01b].

The proof of Theorem 1 builds upon the methods of Trần N. Nam and the first named author. An important new ingredient is the usage of the Milnor operations Q_0, Q_1 , which is a key element of the inductive strategy used in Section 5.

Moreover, the general result proved is stronger than obtained hitherto in the cases of Example 1.1, giving information on the Steenrod operations which are required to hit elements (see Theorem 5.12 below).

Organization: background and references are provided in Section 2 and the results are stated in Section 3, where the standard reduction arguments are explained. The first result, concerning the image of the Steenrod total power, is proved in Section 4; the precise form of the main result is stated and proved in Section 5.

2. BACKGROUND

Fix \mathbb{F}_2 the prime field of characteristic two and write \mathcal{A} for the mod 2 Steenrod algebra. Let $\mathcal{A}(n)$ denote the subalgebra of \mathcal{A} generated by $\{Sq^1, \dots, Sq^{2^n}\}$ for $n \in \mathbb{Z}_+$, where \mathbb{Z}_+ denotes the set of non-negative integers. Recall that Φ denotes the doubling functor on the category \mathcal{U} of unstable modules (respectively unstable algebras, \mathcal{K}) [Sch94].

Let $P_s = \mathbb{F}_2[x_1, \dots, x_s]$ denote the polynomial algebra on s generators of degree 1, equipped with the usual actions of the general linear group $GL_s = GL(s, \mathbb{F}_2)$ and of the Steenrod algebra. The rank s Dickson algebra, D_s , is the algebra of invariants $\mathbb{F}_2[x_1, \dots, x_s]^{GL_s}$, which is an unstable \mathcal{A} -algebra. The underlying algebra is polynomial

$$D_s \cong \mathbb{F}_2[c_{s,0}, \dots, c_{s,s-1}],$$

where $c_{s,i}$ denotes the Dickson invariant of degree $2^s - 2^i$ (for the action of the Steenrod algebra, see [Wil83], [Hun91]).

The Dickson algebra D_s occurs as the unstable algebra $R_s\mathbb{F}_2$, where R_s is the version of the Singer functor used by Lannes and Zarati [LZ87]. This is a functor

$$R_s : \mathcal{U} \rightarrow D_s\text{-}\mathcal{U} \xrightarrow{\text{forget}} \mathcal{U}$$

from unstable modules to the category $D_s\text{-}\mathcal{U}$ of D_s -modules in unstable modules or, by forgetting the D_s -action, to unstable modules. For an unstable module M , there is a canonical inclusion $R_s M \hookrightarrow D_s \otimes M$ in $D_s\text{-}\mathcal{U}$ and the image identifies with the D_s -module generated by the image of $\text{St}_s : \Phi^s M \dashrightarrow P_s \otimes M$, the total Steenrod power map [LZ87] (the dashed arrow indicates that this is linear but not \mathcal{A} -linear in general and the iterated double Φ^s ensures that the degree is preserved). The total Steenrod power is defined recursively using an isomorphism $P_s \cong (P_1)^{\otimes s}$ of unstable algebras, starting from $\text{St}_1(x) = \sum u^{|x|-i} \otimes Sq^i x \in P_1 \otimes M$ and setting $\text{St}_s = \text{St}_1 \circ \text{St}_{s-1}$, interpreted in the appropriate manner. The map St_s takes values in $D_s \otimes M$, in particular is independent of the isomorphism $P_s \cong (P_1)^{\otimes s}$ used in its construction.

The Singer functor R_s has many good properties [LZ87, Pow12]; in particular, it is exact and commutes with tensor products: $R_s(M_1 \otimes M_2) \cong R_s M_1 \otimes_{D_s} R_s M_2$. Moreover, R_s restricts to a functor $R_s : \mathcal{K} \rightarrow D_s \downarrow \mathcal{K}$ from unstable algebras to D_s -algebras in unstable algebras.

An important example is given by the action of R_s on the polynomial algebra P_k :

Proposition 2.1. [LZ87, Section 5.4.7], [HN01b, Theorem 1.2]. *For positive integers k, s , the unstable algebra $R_s P_k$ is isomorphic to the algebra of invariants*

$$R_s P_k \cong P_{k+s}^{\mathrm{GL}_{s,k}}$$

where $\mathrm{GL}_{s,k} \subset \mathrm{GL}_{k+s}$ denotes the subgroup which induces the identity on \mathbb{F}_2^k under the projection $\mathbb{F}_2^{k+s} \cong \mathbb{F}_2^s \oplus \mathbb{F}_2^k \twoheadrightarrow \mathbb{F}_2^k$.

Moreover, there is an isomorphism of algebras

$$(3) \quad R_s P_k \cong \mathbb{F}_2[c_{s,i}, V_s(j) \mid 0 \leq i < s, 1 \leq j \leq k]$$

where $V_s(j) := \mathrm{St}_s x_j$, in particular $|V_s(j)| = 2^s$.

Remark 2.2.

- (i) The identification of the underlying algebra of $R_s P_k$ is not explicit in [LZ87].
- (ii) The group $\mathrm{GL}_{s,k}$ is denoted $\mathrm{GL}_s \bullet \mathbf{1}_k$ in [HN01b].
- (iii) As an element of P_{k+s} , $V_s(j)$ is the Mui invariant $\prod_{y \in \mathbb{F}_2^s} (x_j + y)$, where y ranges over elements of $\mathbb{F}_2^s \subset \mathbb{F}_2^{k+s}$.

For the proof of the main result, it is sufficient to consider the case $s = 3$, hence the following notation is adopted:

Notation 2.3. Write $V(j)$ for $V_3(j) \in R_3 P_k$ for $1 \leq j \leq k$.

Since $R_3 P_k$ is a polynomial algebra on specified generators (3), it is equipped with a length grading:

Notation 2.4. Write $l(\mathbf{m})$ for the length of a monomial \mathbf{m} in $\{c_{s,i}, V(j) \mid 0 \leq i < s, 1 \leq j \leq k\}$ (that is the length of \mathbf{m} as a word).

Definition 2.5. A non-zero element of $R_3 P_k$ is *length homogeneous* if it is the sum of monomials $\sum_i \mathbf{m}_i$ of the same length; its length is $l(\mathbf{m}_i)$ for any monomial appearing.

Recall that the first two Milnor operations are $Q_0 = Sq^1$ and $Q_1 = [Sq^2, Sq^1]$.

Proposition 2.6. [Hun91] *The \mathcal{A} -action on $R_3 P_k$ is determined by*

Sq^0	Sq^1	Sq^2	Sq^3	Sq^4	Sq^5	Sq^6	Sq^7	Sq^8
$c_{3,2}$		$c_{3,1}$	$c_{3,0}$	$c_{3,2}^2$				
$c_{3,1}$	$c_{3,0}$			$c_{3,1}c_{3,2}$	$c_{3,0}c_{3,2}$	$c_{3,1}^2$		
$c_{3,0}$				$c_{3,0}c_{3,2}$		$c_{3,0}c_{3,1}$	$c_{3,0}^2$	
$V(j)$				$V(j)c_{3,2}$		$V(j)c_{3,1}$	$V(j)c_{3,0}$	$V(j)^2$

In particular, the Milnor operations Q_0, Q_1 act trivially on $V(j)$ and $c_{3,0}$, $Q_0 c_{3,2} = Q_1 c_{3,2} = 0$ whereas $Q_1 c_{3,2} = Q_0 c_{3,1} = c_{3,0}$.

Recall that the Milnor operation $Q_i \in \mathcal{A}$ satisfies $Q_i^2 = 0$ and the Margolis cohomology groups of an unstable module M are defined as

$$H^*(M; Q_i) := \mathrm{Ker} Q_i / \mathrm{Im} Q_i$$

with grading inherited from M . Moreover, since $Q_i \in \mathcal{A}$ is primitive with respect to the Hopf algebra structure, it acts as a derivation on unstable algebras.

Notation 2.7. Let \overline{P}_k denotes the augmentation ideal of P_k .

Lemma 2.8. *For $i, k \in \mathbb{Z}_+$,*

- (i) \overline{P}_k is Q_0 -acyclic, that is $\text{Ker}Q_0 = \text{Im}Q_0$ on \overline{P}_k ;
- (ii) $\Phi P_k \subset P_k$ lies in $\text{Ker}Q_i$ and induces a surjection

$$\Phi P_k \cong \mathbb{F}_2[x_j^2 \mid 1 \leq j \leq k] \twoheadrightarrow H^*(P_k; Q_i) \cong \mathbb{F}_2[x_j^2 \mid 1 \leq j \leq k]/(x_j^{2^{i+1}});$$

- (iii) $\text{Ker}(Q_i|_{\overline{P}_k}) = \Phi \overline{P}_k + \text{Im}Q_i \subset \text{Im}Q_0 + \text{Im}Q_i$.

In particular

$$\text{Ker}(Q_0|_{\overline{P}_k}) + \text{Ker}(Q_1|_{\overline{P}_k}) \subset (\text{Im}Sq^1 + \text{Im}Sq^2) = \overline{\mathcal{A}(1)} \overline{P}_k,$$

where $\overline{\mathcal{A}(1)}$ is the augmentation ideal of $\mathcal{A}(1)$.

Proof. The first two points are standard; for instance, to calculate the Margolis cohomology, by the Künneth theorem one reduces to the case $k = 1$, which is elementary.

For (iii), the case of Q_0 is (i). For Q_1 , it follows from (ii) that $\text{Ker}(Q_1|_{\overline{P}_k}) = \Phi \overline{P}_k + \text{Im}(Q_1)$; now $\Phi \overline{P}_k \subset \text{Ker}Q_0 = \text{Im}Q_0$, establishing this case.

Since $Q_0 = Sq^1$ and $Q_1 = [Sq^2, Sq^1]$, the final statement holds. \square

Recall that $F(n)$ denotes the free unstable module on a generator ι_n of degree n , for $n \in \mathbb{Z}_+$ (see e.g. [Ste62], [Sch94]); these modules form a set of projective generators of \mathcal{U} . One identifies $F(0) = \mathbb{F}_2$ and $F(1) = \langle x^{2^i} \mid i = 0, 1, 2, \dots \rangle \subset P_1 \cong \mathbb{F}_2[x]$. Moreover, $F(n)$ is isomorphic to the invariants for the action of the symmetric group by place permutations on $F(1)^{\otimes n}$:

$$F(n) \cong (F(1)^{\otimes n})^{\mathfrak{S}_n}.$$

Hence $F(n)$ embeds in P_n as the submodule generated by the class $\prod_{i=1}^n x_i$; in particular $F(n) \subset P_n^{\mathfrak{S}_n}$, for the action permuting the generators.

Notation 2.9. For M an unstable module, write $\text{St}_s M$ for the image of the linear map $\text{St}_s : \Phi^s M \dashrightarrow D_s \otimes M$.

Remark 2.10. As graded vector spaces, there is a natural isomorphism $\text{St}_s M \cong \Phi^s M$, hence St_s can be considered as an exact functor.

Recall (see [Ste62], for example) that the total Steenrod power is multiplicative: for example, if K is an unstable algebra and $x, y \in K$, then $\text{St}_s(xy) = \text{St}_s(x)\text{St}_s(y)$, where the product is formed in $R_s K \subset P_s \otimes K$.

The following result is stated for $s = 3$ for notational simplicity, but holds for any positive integer s .

Lemma 2.11. *For k a non-negative integer, there is an embedding in $D_3\text{-}\mathcal{U}$:*

$$R_3 F(k) \hookrightarrow R_3 P_k.$$

Moreover, as a graded vector space, $R_3 F(k)$ identifies as the free D_3 -submodule of $D_3 \otimes F(k) \hookrightarrow D_3 \otimes P_k$ generated by $\text{St}_3 F(k)$ and $\text{St}_3 F(k) \subset R_3 P_k$ has a basis of length homogeneous elements.

Proof. The only point which is not immediate from the definitions is that $\text{St}_3 F(k)$ has a basis of length homogeneous elements. For $k = 1$ this is clear: $\text{St}_3 x = V$ (by definition) and the multiplicativity of St_3 gives $\text{St}_3(x^{2^i}) = V^{2^i}$.

For the general case, multiplicativity of St_3 leads to the identification of $\text{St}_3 F(k) \subset R_3 P_k$ with the invariants $(\text{St}_3(F(1))^{\otimes k})^{\mathfrak{S}_k}$. It follows that there is a basis of $\text{St}_3 F(k)$ given by symmetric monomials from

$$(\mathbb{F}_2[V(1), \dots, V(k)])^{\mathfrak{S}_k} \subset \mathbb{F}_2[V(1), \dots, V(k)];$$

in particular, these are length homogeneous. \square

3. REDUCTION ARGUMENTS

For a graded module N , let $N^{>0}$ denote the submodule of elements of positive degree. The main result of the paper (Theorem 1 of the introduction) can be stated as follows:

Theorem 3.1. *For an unstable module M and an integer $s \geq 3$, the natural map*

$$(R_s M)^{>0} \rightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes M)$$

induced by $R_s M \hookrightarrow D_s \otimes M \hookrightarrow P_s \otimes M$ and passage to \mathcal{A} -indecomposables is zero.

The following standard argument reduces to the case $s = 3$:

Proposition 3.2. *Suppose that $(R_3 M)^{>0} \rightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} (P_3 \otimes M)$ is zero for each unstable module M , then*

$$(R_s M)^{>0} \rightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes M)$$

is zero for each unstable module M and all $s \geq 3$.

Proof. For $s \geq 3$, by construction (see [LZ87]) there is a natural inclusion $R_s M \hookrightarrow R_3 R_{s-3} M$ which fits into the commutative diagram

$$\begin{array}{ccc} R_s M & \longrightarrow & \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes M) \\ \downarrow & & \uparrow \\ R_3 R_{s-3} M & \longrightarrow & \mathbb{F}_2 \otimes_{\mathcal{A}} (P_3 \otimes R_{s-3} M), \end{array}$$

where the upwards arrow is induced by $R_{s-3} M \hookrightarrow P_{s-3} \otimes M$ and the isomorphism $P_s \cong P_3 \otimes P_{s-3}$. The lower horizontal arrow is the natural morphism $R_3 N \rightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} (P_3 \otimes N)$ with $N = R_{s-3} M$. By hypothesis, this is zero on positive degree elements, hence so is the upper. \square

Notation 3.3. For s a positive integer and M an unstable module, write $\widetilde{R}_s M$ for $\overline{D}_s \otimes_{D_s} R_s M$.

Recall [LZ87] that, for s a positive integer, there is a canonical surjection in \mathcal{U} :

$$(4) \quad R_s M \twoheadrightarrow \Phi^s M \cong \mathbb{F}_2 \otimes_{D_s} R_s M.$$

Lemma 3.4. *For s a positive integer and M an unstable module,*

- (i) $\widetilde{R}_s M$ is the kernel of the surjection (4);
- (ii) there is a natural isomorphism of graded vector spaces $R_s M \cong \widetilde{R}_s M \oplus \text{St}_s M$;
- (iii) the functor $\widetilde{R}_s : \mathcal{U} \rightarrow \mathcal{U}$ is exact and commutes with direct sums.

Proof. The first statement follows by applying the exact functor $-\otimes_{D_s} R_s M$ ($R_s M$ is free as a D_s -module) to the short exact sequence of D_s -modules:

$$0 \rightarrow \overline{D}_s \rightarrow D_s \rightarrow \mathbb{F}_2 \rightarrow 0.$$

The remaining statements are straightforward. \square

The following is an immediate consequence of the fact that P_s is free as a D_s -module:

Lemma 3.5. *For s a positive integer and M an unstable module, the canonical inclusion $R_s M \hookrightarrow D_s \otimes M$ induces an inclusion*

$$P_s \otimes_{D_s} R_s M \hookrightarrow P_s \otimes M.$$

As a P_s -module, the image is free on $\text{St}_s M$.

The natural map $R_s M \rightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes M)$ factors:

$$\begin{array}{ccc} R_s M & \longrightarrow & \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes_{D_s} R_s M) \\ & \searrow & \downarrow \\ & & \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes M) \end{array}$$

and restricts to a linear map $\Phi^s M \cong \text{St}_s M \dashrightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes M)$.

Remark 3.6. Since $(\text{St}_s M)^{>0} = \text{St}_s(M^{>0})$, parentheses can be omitted.

Theorem 3.1 follows (using Lemma 3.4) from the refined statement:

Theorem 3.7. *For an unstable module M and an integer $s > 2$,*

- (i) $\text{St}_s M^{>0} \dashrightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes M)$ is zero;
- (ii) the composite morphism $\widetilde{R}_s M \hookrightarrow R_s M \rightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes_{D_s} R_s M)$ is zero.

Remark 3.8. By Proposition 3.2, it suffices to treat the case $s = 3$. For this, a more refined result is stated in Theorem 5.12.

The theorem can be proved by considering the projective generators of \mathcal{U} :

Lemma 3.9. *Theorem 3.7 holds for all unstable modules M if and only if it holds for the free unstable module $F(n)$, for any non-negative integer n .*

Proof. This is a formal consequence of the fact that the set of all $F(n)$, n a non-negative integer, forms a set of projective generators of \mathcal{U} and that $\text{St}_3, \widetilde{R}_3$ are exact and commute with direct sums, by Lemma 3.4. \square

4. THE IMAGE OF St_s

Recall that, for M an unstable module, St_3 denotes the linear map

$$\Phi^3 M \dashrightarrow D_3 \otimes M \subset P_3 \otimes M.$$

The aim of this section is to prove Proposition 4.5 below, which corresponds to the first statement of Theorem 3.7.

Lemma 4.1. *For $x \in M$, where M is an unstable module, Q_0, Q_1, Sq^2 act trivially on $\text{St}_3(x)$.*

Proof. It is straightforward to reduce to the universal example $x = \iota_n \in F(n)$, the fundamental class of the free unstable module $F(n)$ (see Section 2). Now $F(n) \hookrightarrow P_n = \mathbb{F}_2[x_1, \dots, x_n]$ under the map induced by $\iota_n \mapsto \prod_{i=1}^n x_i$, so that $\text{St}_3(\iota_n) \in \text{St}_3(F(n)) \subset R_3 P_n$; here one has $\text{St}_3(\iota_n) = \prod_{j=1}^n V(j)$. Since Sq^1, Sq^2 (and hence Q_1) act trivially on $V(j)$ by Proposition 2.6, the result follows from the Cartan formula. \square

Consider the subalgebra $\mathcal{A}(0) \subset \mathcal{A}$ generated by Q_0 , which identifies with the exterior algebra $\Lambda(Q_0)$.

Lemma 4.2. *As an $\mathcal{A}(0)$ -module:*

$$D_3 \cong (\mathbb{F}_2[c_{3,2}], Q_0 c_{3,2} = 0) \otimes (\mathbb{F}_2[c_{3,1}, c_{3,0}], Q_0 c_{3,1} = c_{3,0}).$$

Hence

- (i) $H_*(D_3; Q_0) \cong \mathbb{F}_2[c_{3,2}, c_{3,1}^2]$;
- (ii) *a monomial in $\{c_{3,0}, c_{3,1}, c_{3,2}\}$ of D_3 lies in $\text{Ker} Q_0$ if and only if it has the form $c_{3,2}^{i_2} c_{3,1}^{2n} c_{3,0}^t$ for $i_2, n, t \in \mathbb{Z}_+$; if $t > 0$ then it lies in $\text{Im} Q_0$.*

Proof. Straightforward. \square

Notation 4.3. Index the monomial basis of D_3 by $I = (i_2, i_1, i_0) \in \mathbb{Z}_+^3$ with

$$(5) \quad c^I = c^{i_2, i_1, i_0} := c_{3,2}^{i_2} c_{3,1}^{i_1} c_{3,0}^{i_0}.$$

Lemma 4.4. *Let M be an $\mathcal{A}(0)$ -module and write an element $y \in \overline{D}_3 \otimes M$ as $y = \sum c^I \otimes y_I$, where c^I runs over the monomial basis (5) of \overline{D}_3 .*

Then y lies in $\text{Ker} Q_0$ if and only if both

- (i) $Q_0 y_{i_2, 2u, t+1} = y_{i_2, 2u+1, t}$ for all $(i_2, u, t) \in \mathbb{Z}_+^3$ and
- (ii) $Q_0 y_{i_2, 2u, 0} = 0$.

If these conditions are satisfied,

$$y = \sum c^{i_2, 2u, 0} \otimes y_{i_2, 2u, 0} + \sum Q_0(c^{i_2, 2u+1, t} \otimes y_{i_2, 2u, t+1}).$$

Proof. Clearly $Q_0 y = \sum \{(Q_0 c^I) \otimes y_I + c^I \otimes Q_0 y_I\}$. The result follows by identifying coefficients in the expansion in terms of the monomial basis of D_3 . \square

Proposition 4.5. *For M an unstable module, $\text{St}_3 M^{>0} \subset \overline{\mathcal{A}}(P_3 \otimes M)$. More precisely, $\text{St}_3(x) \in \text{Im} Sq^1 + \text{Im} Sq^{4|x|}$ for $x \in M^{>0}$.*

Proof. As in Lemma 4.1, one reduces to the universal example $\iota_n \in F(n)$: it suffices to show that $\text{St}_3 \iota_n \in \overline{\mathcal{A}}(P_3 \otimes F(n))$ for each $0 < n \in \mathbb{Z}_+$.

Lemma 4.1 implies that $\text{St}_3 \iota_n \in \text{Ker} Q_0$; Lemma 4.4 shows that it suffices to consider the terms

$$c_{3,2}^{i_2} c_{3,1}^{2u} \otimes y_{i_2, 2u}$$

which appear in the expansion of $\text{St}_3 \iota_n$ (for simplicity, writing $y_{i_2, 2u}$ for $y_{i_2, 2u, 0}$). By Lemma 4.4, both $y_{i_2, 2u}$ and $c_{3,2}^{i_2} c_{3,1}^{2u}$ lie in $\text{Ker} Q_0$.

Clearly $y_{0,0} = (Sq_0)^3 \iota_n \in F(n)$ (recall that $Sq_0(x) := Sq^{|x|}x$); in the other cases, $c_{3,2}^{i_2} c_{3,1}^{2u}$ has positive degree so that there exists $a_{i_2, 2u} \in P_3$ such that $Q_0 a_{i_2, 2u} = c_{3,2}^{i_2} c_{3,1}^{2u}$, by Lemma 2.8. Thus $Q_0(a_{i_2, 2u} \otimes y_{i_2, 2u}) = c_{3,2}^{i_2} c_{3,1}^{2u} \otimes y_{i_2, 2u}$ in $P_3 \otimes F(n)$.

This establishes that, for $x \in M^{>0}$, $\text{St}_3(x) \equiv 1 \otimes (Sq_0)^3(x) = (Sq_0)^3(1 \otimes x) \pmod{\text{Im} Q_0}$ in $P_3 \otimes M$. Since $(Sq_0)^3(x) = Sq^{4|x|}(Sq_0)^2(x)$, the final statement follows. \square

Corollary 4.6. *For an integer $s \geq 3$ and an unstable module M ,*

$$\mathrm{St}_s M^{>0} \subset \overline{\mathcal{A}}(P_s \otimes M).$$

Proof. Use a reduction argument similar to the proof of Proposition 3.2. \square

5. PROOF OF THE MAIN THEOREM

To complete the proof of Theorem 3.7, by combining Lemma 3.9 with Proposition 4.5, it suffices to show that

$$\widetilde{R_3 F}(n) \subset \overline{\mathcal{A}}(P_3 \otimes_{D_3} R_3 F(n))$$

for each $n \in \mathbb{Z}_+$.

Recall that $P_3 \otimes_{D_3} R_3 M \cong P_3 \otimes \mathrm{St}_3 M$ as a P_3 -module (forgetting the \mathcal{A} -action). The inclusion $\overline{P}_3 \hookrightarrow P_3$ of D_3 -modules induces

$$\overline{P}_3 \otimes \mathrm{St}_3 M \cong \overline{P}_3 \otimes_{D_3} R_3 M \hookrightarrow P_3 \otimes_{D_3} R_3 M \cong P_3 \otimes \mathrm{St}_3 M.$$

Lemma 5.1. *For M an unstable module, there is a natural isomorphism of $\mathcal{A}(1)$ -modules*

$$P_3 \otimes_{D_3} R_3 M \cong P_3 \otimes \mathrm{St}_3 M,$$

where $\mathcal{A}(1)$ acts trivially on $\mathrm{St}_3 M$. This restricts to an isomorphism of $\mathcal{A}(1)$ -modules $\overline{P}_3 \otimes_{D_3} R_3 M \cong \overline{P}_3 \otimes \mathrm{St}_3 M$.

Hence

$$(\overline{P}_3 \otimes \mathrm{St}_3 M) \cap (\mathrm{Ker} Q_0 + \mathrm{Ker} Q_1) \subset (\mathrm{Im} Sq^1 + \mathrm{Im} Sq^2) \subset P_3 \otimes_{D_3} R_3 M.$$

Proof. Lemma 4.1 implies that $\mathrm{St}_3 M \subset R_3 M$ is a trivial $\mathcal{A}(1)$ -submodule. The P_3 -module structure of $P_3 \otimes_{D_3} R_3 M$ then leads to the isomorphism of P_3 -modules

$$P_3 \otimes \mathrm{St}_3 M \rightarrow P_3 \otimes_{D_3} R_3 M$$

which is $\mathcal{A}(1)$ -linear. The case of the restriction to \overline{P}_3 is straightforward.

The final statement follows from Lemma 2.8. \square

Recall that $\widetilde{R_3 F}(n) \subset R_3 F(n) \subset R_3 P_n$ and that $R_3 F(n)$ is the free D_3 -module on $\mathrm{St}_3 F(n)$. Moreover, Lemma 2.11 implies that $\mathrm{St}_3 F(n) \subset R_3 P_n$ has a basis given by length homogeneous polynomials in $\mathbb{F}_2[V(1), \dots, V(n)]$.

Lemma 5.2. *For n a non-negative integer, $\widetilde{R_3 F}(n) \subset R_3 F(n)$ has a basis of elements of the form*

$$(6) \quad c^I \mathrm{St}_3 y_i$$

where $y_i \in F(n)$ ranges over a basis such that $\mathrm{St}_3 y_i$ is length homogeneous and $c^I \neq 1$ ranges over the monomial basis.

Proof. Straightforward. \square

Notation 5.3. A term of an element of $\widetilde{R_3 F}(n)$ is a basis element (as in equation (6) of Lemma 5.2) which appears with non-zero coefficient.

To prove Theorem 3.7, it is sufficient to consider the basis elements (6) of Lemma 5.2. The inductive proof is based upon the submonomial of c^I in the generators $c_{3,2}$ and $c_{3,1}$; the behaviour is analysed using Lemmas 5.4 and 5.5 below.

Lemma 5.4. *Let $\mathfrak{m} = c^I \mathrm{St}_3 y \in \widetilde{R_3 F}(n)$ where $c^I \neq 1$, $\mathrm{St}_3 y$ is length homogeneous and let $v \in P_3$. If $i_1 i_2 \equiv 0 \pmod{2}$ then $\mathfrak{m} v^2 \in (\mathrm{Im} Sq^1 + \mathrm{Im} Sq^2) \subset P_3 \otimes_{D_3} R_3 M$.*

Proof. If $i_1 \equiv 0 \pmod{2}$, then $Q_0 \mathbf{m} = 0$ and if $i_2 \equiv 0 \pmod{2}$, then $Q_1 \mathbf{m} = 0$, by Proposition 2.6. Since Q_i ($i \in \{0, 1\}$) is a derivation, if $Q_i x = 0$, then $Q_i x v^2 = 0$; hence, under the hypothesis, $\mathbf{m} v^2 \in \text{Ker} Q_0 + \text{Ker} Q_1$. The result thus follows from Lemma 5.1. \square

This result is complemented by the following special case of [HN01a, Lemma B].

Lemma 5.5. *The element $c_{3,2}c_{3,1} \in D_3$ lies in $\text{Im} Sq^1 + \text{Im} Sq^2 \subset P_3$.*

Lemmas 5.4 and 5.5 motivate the following:

Definition 5.6. For $\mathbf{m} = c^I \text{St}_3 y$ as above:

- (i) the standard form for \mathbf{m} is
- (7)
$$\mathbf{m} = (c_{3,2}c_{3,1})^{2^f-1} (c_{3,2}^{j_2} c_{3,1}^{j_1})^{2^f} c_{3,0}^{i_0} \text{St}_3 y,$$

where $f \in \mathbb{Z}_+$ is determined by $j_1 j_2 \equiv 0 \pmod{2}$;
- (ii) the fullness of \mathbf{m} , $f(\mathbf{m}) \in \mathbb{Z}_+$, is the integer f appearing in the standard form;
- (iii) \mathbf{m} is full if $j_1 = 0 = j_2$.

Remark 5.7. The significance of the notion of *fullness* is suggested by Lemma 5.4 and is important in the primary induction of the proof of the main theorem below. A secondary argument uses the notion of *length* inherited from $R_3 P_n$ (see Definition 2.5 and Lemma 2.11).

Remark 5.8. The length argument uses Lemma 5.10 below (which is inspired by [HN01b, Lemma 3.4]), which allows monomials which satisfy

$$l(\mathbf{m}) \not\equiv 2^{f(\mathbf{m})} - 1 \pmod{2^{f(\mathbf{m})}}$$

to be treated by using an induction upon the fullness.

The remaining case is studied by using Lemma 5.11 (which is inspired by [HN01b, Lemma 3.5]); this reduces to terms of smaller fullness or terms which can be treated by Lemma 5.10.

The following elementary observation is used:

Lemma 5.9. *Let f and ℓ be non-negative integers. Then $\ell \equiv 2^f - 1 \pmod{2^f}$ if and only if*

$$\binom{\ell}{2^i} \equiv 1 \pmod{2},$$

for each $0 \leq i < f$.

Lemma 5.10. *For $\mathbf{m} = c^I \text{St}_3 y \in \widetilde{R}_3 F(n)$ with $\text{St}_3 y$ length homogeneous of length $l(\mathbf{m})$, and $f(\mathbf{m}) = f > 0$, suppose that, for some $0 \leq i < f$,*

$$\binom{l(\mathbf{m})}{2^i} \equiv 0 \pmod{2}.$$

Then, $\mathbf{m} c_{3,2}^{-2^i} \in \widetilde{R}_3 F(n)$ and, for any element $v \in P_3$, there is an equality in $P_3 \otimes_{D_3} R_3 F(n)$:

$$Sq^{4 \cdot 2^i} (\mathbf{m} c_{3,2}^{-2^i} v^{2^f}) = \mathbf{m} v^{2^f} + \sum_l \mathfrak{s}_l v_l^{2^f}$$

where $v_l \in P_3$ and $\mathfrak{s}_l \in R_3 F(n)$ is an element of the form $c^{I_l} \text{St}_3(y_l)$ such that

- (i) $\mathfrak{s}_l \in \widetilde{R}_3 F(n)$;

(ii) $f(\mathfrak{s}_l) \leq i < f$.

Proof. The fact that $\mathfrak{m}c_{3,2}^{-2^i} \in \widetilde{R_3F}(n)$ is clear. The remainder of the argument uses the action of the Steenrod algebra (given in Proposition 2.6) together with the Cartan formula. In particular, Sq^4 acts on the generators of R_3P_n as multiplication by $c_{3,2}$.

A monomial appearing in the expansion of $Sq^{4 \cdot 2^i}(\mathfrak{m}c_{3,2}^{-2^i}v^{2^f})$ can only have fullness $> i$ by the application of Sq^4 to 2^i generators. The hypothesis on the binomial coefficient ensures that this contributes the term $\mathfrak{m}v^{2^f}$.

The remaining terms have fullness $\leq i$ and can be written in the required form $\mathfrak{s}_lv_l^{2^f}$. The fact that $\mathfrak{s}_l \in \widetilde{R_3F}(n)$ is clear, since it is impossible to destroy a contribution from $\overline{D_3}$. \square

Lemma 5.11. *Let $\mathfrak{m} = c^I \text{St}_3 y \in \widetilde{R_3F}(n)$ with $\text{St}_3 y$ length homogeneous and \mathfrak{m} full with $f(\mathfrak{m}) = f$.*

Suppose that $l(\mathfrak{m}) \equiv 2^f - 1 \pmod{2^f}$ and consider a Steenrod operation $\vartheta \in \mathcal{A}$ such that $|\vartheta| \leq 2^{f+1}$.

For \mathfrak{t} a term of $\vartheta\mathfrak{m}$,

- (i) *if $l(\mathfrak{t}) = l(\mathfrak{m})$, then $f(\mathfrak{t}) < f$;*
- (ii) *if $l(\mathfrak{t}) > l(\mathfrak{m})$, then $l(\mathfrak{t}) \not\equiv 2^f - 1 \pmod{2^f}$.*

In particular, one of the following holds:

- (i) $f(\mathfrak{t}) < f$;
- (ii) $f(\mathfrak{t}) \geq f$ and $l(\mathfrak{t}) \not\equiv 2^{f(\mathfrak{t})} - 1 \pmod{2^{f(\mathfrak{t})}}$.

Proof. The hypothesis that \mathfrak{m} is full of fullness f means that

$$\mathfrak{m} = (c_{3,2}c_{3,1})^{2^f-1}c_{3,0}^{i_0}\text{St}_3 y.$$

In the case $l(\mathfrak{t}) = l(\mathfrak{m})$, \mathfrak{t} arises from the action of the operation ϑ upon the factor $(c_{3,2}c_{3,1})^{2^f-1}$ of \mathfrak{m} since Steenrod operations increase the length when operating non-trivially on the other terms (this is where fullness is used). It follows by inspection that $f(\mathfrak{t}) < f$.

In the remaining case, write $l(\mathfrak{m}) = 2^f - 1 + k2^f$, for some $k \in \mathbb{Z}_+$. Then, by inspection of the action given by Proposition 2.6,

$$l(\mathfrak{m}) = 2^f - 1 + k2^f < l(\mathfrak{t}) \leq 2^f - 1 + k2^f + |\vartheta|/4 \leq 2^f - 1 + k2^f + 2^{f-1}.$$

The conclusion follows by elementary arithmetic. \square

Theorem 3.7 is implied by the following, more precise, result. The main case of interest is when $v = 1$ in the statement; the inductive proof requires the stronger statement.

Theorem 5.12. *Let $\mathfrak{m} = c^I \text{St}_3 y \in \widetilde{R_3F}(n)$ be a basis element with $f(\mathfrak{m}) = f$ and let $v \in P_3$. Then*

- (i) *if \mathfrak{m} is full (hence $f > 0$)*

$$\mathfrak{m}v^{2^{f+1}} \in \overline{\mathcal{A}(f)}(P_3 \otimes_{D_3} R_3F(n));$$

- (ii) *otherwise*

$$\mathfrak{m}v^{2^{f+1}} \in \overline{\mathcal{A}(f+1)}(P_3 \otimes_{D_3} R_3F(n)).$$

Proof. The result is proved by increasing induction upon f .

The initial case of the induction for $f = 0$ (only the non-full case occurs), is established by Lemma 5.4 since, by definition of fullness, the hypothesis $i_1 i_2 \equiv 0 \pmod{2}$ is satisfied.

The initial case of the induction for the full case is for $f = 1$; here one can write:

$$\mathbf{m}v^4 = c_{3,2}c_{3,1}\mathbf{m}'v^4$$

and Sq^1, Sq^2 act trivially on $\mathbf{m}'v^4$ (since \mathbf{m}' contains neither $c_{3,2}$ nor $c_{3,1}$). The result therefore follows from Lemma 5.5 and the Cartan formula.

There are two inductive steps to consider. In order to present a unified proof of the final step, the induction first treats the non-full case with $f(\mathbf{m}) = f$ and then the full case with $f(\mathbf{m}) = f + 1$.

- (i) In the non-full case with $f(\mathbf{m}) = f$, one can write

$$\mathbf{m}v^{2^{f+1}} = (c_{3,2}^{j_2}c_{3,1}^{j_1}v^2)^{2^f}\mathbf{m}_{\text{full}}$$

where \mathbf{m}_{full} is full with $f(\mathbf{m}_{\text{full}}) = f$, $j_1 + j_2 > 0$ and $j_1 j_2 \equiv 0 \pmod{2}$. Lemma 5.4 implies that $c_{3,2}^{j_2}c_{3,1}^{j_1}v^2 \in (\text{Im}Sq^1 + \text{Im}Sq^2) \subset \overline{P}_3$, say is $Sq^1u_1 + Sq^2u_2$, so that $(c_{3,2}^{j_2}c_{3,1}^{j_1}v^2)^{2^f} = Sq^{2^f}u_1^{2^f} + Sq^{2^{f+1}}u_2^{2^f}$.

Hence it is sufficient to consider elements of the form

$$(Sq^{2^f}u_1^{2^f} + Sq^{2^{f+1}}u_2^{2^f})\mathbf{m}_{\text{full}},$$

where $u_1, u_2 \in \overline{P}_3$. For such elements, see below.

- (ii) In the full case with $f(\mathbf{m}) = f + 1$, first observe that it suffices to prove the case for $v = 1$, since if $\mathbf{m} \in \text{Im}(Sq^1, \dots, Sq^{2^{f+1}})$, the Cartan formula shows that $\mathbf{m}v^{2^{f+2}}$ is also.

Now write

$$\mathbf{m} = (c_{3,2}c_{3,1})^{2^f}\mathbf{m}_{\text{full}}$$

where \mathbf{m}_{full} is full with $f(\mathbf{m}_{\text{full}}) = f$. By Lemma 5.5, $c_{3,2}c_{3,1}$, considered as an element of P_3 , lies in $\text{Im}Sq^1 + \text{Im}Sq^2$. Hence $(c_{3,2}c_{3,1})^{2^f}$ lies in $\text{Im}Sq^{2^f} + \text{Im}Sq^{2^{f+1}}$, thus it is sufficient to consider elements of the form:

$$(Sq^{2^f}u_1^{2^f} + Sq^{2^{f+1}}u_2^{2^f})\mathbf{m}_{\text{full}},$$

where $u_1, u_2 \in \overline{P}_3$. This expression has the same form as in the non-full case.

The indexing was chosen so that $f(\mathbf{m}_{\text{full}}) = f$ in both cases. To establish the inductive steps, it is sufficient to show, for $u_1, u_2 \in \overline{P}_3$ and $f(\mathbf{m}_{\text{full}}) = f$, that

$$(Sq^{2^f}u_1^{2^f} + Sq^{2^{f+1}}u_2^{2^f})\mathbf{m}_{\text{full}} \in \text{Im}(Sq^1, \dots, Sq^{2^{f+1}}) \subset P_3 \otimes_{D_3} R_3F(n).$$

(Note that the appropriate subalgebra of \mathcal{A} is $\mathcal{A}(f+1)$ in both cases, by the choice of indexing.)

Hence consider elements of the form

$$(Sq^\delta u)^{2^f}\mathbf{m}_{\text{full}} = Sq^{2^{f-1+\delta}}(u^{2^f})\mathbf{m}_{\text{full}}$$

for $u \in \overline{P}_3$ and $\delta \in \{1, 2\}$ and $f(\mathbf{m}_{\text{full}}) = f$.

If $l(\mathbf{m}_{\text{full}}) \not\equiv 2^f - 1 \pmod{2^f}$, then Lemma 5.10, using $Sq^{2^{i+2}}$ for the appropriate i , $0 \leq i < f$, allows reduction to terms earlier in the inductive scheme, since

- (i) $i + 2 < f + 2$, so that $i + 2 \leq f + 1$ and hence $Sq^{2^{i+2}} \in \mathcal{A}(f + 1)$;

- (ii) the terms $\mathfrak{s}_l v_l^{2^f}$ (in the notation of Lemma 5.10) which occur have fullness $f(\mathfrak{s}_l) < f$, hence are treated by the inductive hypothesis.

Finally, if $l(\mathbf{m}_{\text{full}}) \equiv 2^f - 1 \pmod{2^f}$, consider

$$Sq^{2^{f-1}+\delta}(u^{2^f} \mathbf{m}_{\text{full}})$$

using the Cartan formula and Lemma 5.11 to understand the terms which occur from applying a Steenrod square to \mathbf{m}_{full} , which are of the form $Sq^j \mathbf{m}_{\text{full}}$ with $j \leq 2^{f+1}$. (In fact, only $Sq^j \mathbf{m}_{\text{full}}$ with $j \in \{2^f, 2^{f+1}\}$ have to be considered, but this precision is not required.) Note that the operation $Sq^{2^{f-1}+\delta}$ lies in $\mathcal{A}(f+1)$.

By the final statement of Lemma 5.11, the terms which arise either have fullness $< f$ or can be treated as above by using Lemma 5.10.

This completes the proof of the inductive steps. \square

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